**Students' Difficulties with Proof[[1]](#footnote-1)**

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Proof is a notoriously difficult mathematical concept for students. Empirical studies have shown that many students emerge from proof-oriented courses such as high school geometry ([Senk](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Sen1), 1985), introduction to proof ([Moore](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Mo), 1994), real analysis ([Bills and Tall](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Bi), 1998), and abstract algebra ([Weber](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#We1), 2001) unable to construct anything beyond very trivial proofs. Furthermore, most university students do not know what constitutes a proof ([Recio and Godino](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Rec), 2001) and cannot determine whether a purported proof is valid ([Selden and Selden](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Sel1), 2003).

**What is proof and what is its role in mathematics?**

Many mathematicians and mathematics teachers would consider the answers to this question straightforward. The traditional view is that a mathematical proof is a formal and logical line of reasoning that begins with a set of axioms and moves through logical steps to a conclusion ([Griffiths](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof%22%20%5Cl%20%22Gr), 2000, p. 2). And, the purpose of proving a theorem is to establish its mathematical certainty. A proof confirms truth for a mathematician the way experiment or observation does for the natural scientists ([Griffiths](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Gr), 2000, p. 2). Such views are commonly held by mathematics teachers and are passed along to our students. However, many mathematics educators and some mathematicians believe that proofs are much more than this.

Fields Medalist [William Thurston](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Th)(1994) argues that it is important to distinguish between formal proofs and proofs that mathematicians actually construct. In the latter case, many routine calculations and logical manipulations are suppressed. Such omissions are not due to carelessness; rather this is done because proofs would be impossibly long if every logical detail were included. [Renz](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Ren) (1981) has estimated that if one proceeded to prove the Pythagorean theorem using only the axioms and rules of inference allowed in Euclid's *Elements*, the proof would be over 80 pages long.

[Davis and Hersh](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Da) (1981) argue that it is probably impossible to define precisely what type of argument will be accepted as a valid proof by the mathematical community. Of course, there are some aspects of proof that distinguish it from other types of arguments. For example, proofs about a concept must use the concept's definition and must proceed deductively, as opposed to examining prototypical cases or giving an intuitive argument. And if a result is incorporated in a proof, that result must be accepted by the mathematical community ([Tall](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Ta1), 1989). Beyond this, some mathematics educators argue that whether or not an argument is accepted as a proof depends not only on its logical structure, but also on how convincing the argument is ([Hanna](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Han2), 1991).

At different places in the mathematics education literature, a proof has been defined as an argument that convinces an enemy ([Mason, Burton, and Stacey](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Mas), 1982), an argument that convinces a mathematician who knows the subject ([Davis and Hersh](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Da), 1981), or an argument that suffices to convince a reasonable skeptic ([Volmink](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Vo), 1990). Others, who focus on the social and contextual nature of proof, offer the following relativist descriptions: We call proof an explanation accepted by a given community at a given time ([Balacheff](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Ba), 1987, translated from French). An argument becomes a proof after the social act of accepting it as a proof ([Manin](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Man), 1977). Many mathematics educators believe that focusing exclusively on the logical nature of proof can be harmful to students' development. Such a narrow view leads students to focus on logical manipulations rather than on forming and understanding convincing explanations for why a statement is true ([Alibert and Thomas](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Ali2), 1991).

Mathematics educators and mathematicians believe that establishing the veracity of a statement is only one of many reasons for constructing or presenting a proof. Besides convincing, mathematics educators have proposed a number of alternative purposes of proof. For example,

* *Explanation.* By examining a proof, a reader can understand *why* a certain statement is true. Many mathematics educators argue that explanation should be the primary purpose of proof in the mathematics classroom (e.g., [Hanna](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Han1), 1990; [Hersh](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#He), 1993).
* *Systemization.* One can use proofs to organize previously disparate results into a unified whole. By organizing a system deductively, one can also uncover arguments that may be fallacious, circular, or incomplete ([de Villiers](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#DeV), 1990).
* *Communication.* The language of proof can be used to communicate and debate ideas with other students and mathematicians (e.g., [de Villiers](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#DeV), 1990; [Knuth](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Kn), 2002).
* *Discovery of new results.* By exploring the logical consequences of definitions and an axiomatic system, new models or theories can be developed ([de Villiers](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#DeV), 1990).
* *Justification of a definition.* One can show that a definition is adequate to capture the intuitive essence of a concept by proving that all of the concept's essential properties can be derived from the proposed definition ([Weber](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#We2), 2002a).
* *Developing intuition.* By examining the logical entailments of a concept's definition, one can sometimes develop a conceptual and intuitive understanding of the concept that one is studying ([Pinto and Tall](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Pi), 1999).
* *Providing autonomy.* Teaching students how to prove can allow them to independently construct and validate new mathematical knowledge ([Yackel and Cobb](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Ya), 1996).

Students who believe that proofs are used solely to establish the certainty of mathematical statements find proofs of seemingly obvious results (e.g., 1+1 = 2 using the Peano postulates) to be pedantic ([Harel](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Har1), 1998; [Weber](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#We2), 2002a). On the other hand, an awareness of the varied uses of proof may improve students' appreciation for the role of proof in mathematics.

**What difficulties do students have with proofs?**

**2.1 Students' conceptions of proof**

Consider the following proof that a student recently produced in the introductory proof course that I am currently teaching:

Show that for every odd integer *n*, *n*2 - 1 is divisible by 8.

12 - 1 = 0 which is divisible by 8. 32 - 1 = 8 which is divisible by 8. 52 - 1 = 24 which is divisible by 8. And so on. Therefore, if *n* is odd, *n*2 - 1 is divisible by 8.

Variations of this proof are surprisingly common. They illustrate one of students' most ubiquitous difficulties with the concept of proof: Students often believe that non-deductive arguments constitute a proof. Below are some common student beliefs about what constitutes a mathematical proof. A comprehensive taxonomy of such beliefs is given in [Harel and Sowder](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof%22%20%5Cl%20%22Har2) (1998).

* *Ritual.* An argument is a proof if and only if it is in accordance with a specific mathematical convention. For instance, many pre-service teachers believe a geometry argument must be in a two-column format to be a proof ([Martin and Harel](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Marti), 1989).
* *Authoritative.* An argument is a proof if it is presented by or approved by an established authority, such as a teacher or a famous mathematician.
* *Inductive.* Checking that a general statement holds for one example, or perhaps several examples, is sufficient to demonstrate its veracity. The above faulty proof is an example.
* *Perceptual.* By way of an appropriate diagram, one can visually demonstrate that a certain property holds ([Harel and Sowder](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Har2), 1998). For instance, to prove that the sequence 1/*n* converges, some students will draw a diagram illustrating that as n grows large, the terms of 1/*n* will become arbitrarily close to zero.

Recently, research has focused on why students may possess these invalid beliefs about proof. [Recio and Godino](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Rec) (2001) note that many such invalid proof techniques would be appropriate in non-mathematical domains. For instance, drawing a general conclusion by examining many specific cases is entirely appropriate in the social sciences. [Alcock and Simpson](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Alc) (2002) observe that reasoning about a concept using a prototypical example is common in our everyday experience, but in the realm of formal mathematics, one must reason using the concept's definition.

**2.2 Inadequate cognitive development**

Two Dutch researchers, Dina and Pierre van Hiele, proposed a learning cycle (now known as the van Hiele levels) through which students may progress as they learn Euclidean geometry:

* *Level 0: Visualization.* Students can recognize a geometric figure as an entity (e.g., a square), but cannot recognize properties of this figure (e.g., a right angle).
* *Level 1: Analysis.* Students can recognize components and properties of a figure. However, students cannot see relationships between properties and figures, nor can they define a figure in terms of its properties. For instance, students at this stage may observe that all rectangles have four right angles, but they would not realize that this entailed a square was a rectangle.
* *Level 2: Informal deduction.* Students can recognize interrelationships between figures and properties and they can justify these relationships informally. Such students might recognize that a square is a rectangle since it had all the properties of a rectangle, but be unable to produce arguments starting with unfamiliar premises. For example, such students could not construct trivial proofs about objects that were unfamiliar to them even if they knew how that object was defined.
* *Level 3: Deduction.* Students can reason about geometric objects using their defined properties in a deductive pattern. They can use an axiom system to construct proofs. Students at this stage could construct the types of proofs that one would find in a typical high school geometry course (e.g., isosceles triangles have two congruent angles).
* *Level 4: Rigor.* Students can compare different axiom systems. Geometry is seen as an abstract rigorous field. (See [Teppo](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Te), 1991).

The van Hieles postulated that (a) students must progress through each of these stages, i.e. they cannot ’skip a level, and (b) instruction targeted for students at a higher level will not be comprehensible for students at a lower level. [Senk](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Sen2) (1989) found strong empirical support for the van Hieles' claim. In a large scale study, Senk demonstrated that by determining a student's van Hiele level at the beginning of a high school geometry course, one could very accurately predict that students' proof-writing ability at the end of the course. Senk also found that many students enter high school geometry with a low Van Hiele level of understanding, and suggests that this may be why geometry gives high school students so much difficulty ([Senk](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Sen1) 1985, [1989](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Sen2)).

A lack of cognitive development may also prevent college students from understanding the concept of proof. Piaget claims that students will be unable to discern or construct deductive arguments until they have reached what he calls a formal operational stage of cognitive development. [Ausubel and his colleagues](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Au) (1968) investigated college students' stages of cognitive development and found that just 22% of college students had achieved a formal operational stage of development. These findings have led [Tall](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Ta2) (1991) and others to observe that many university students may (at least initially) be unable to understand deductive proofs.

**2.3 Notational difficulties**

Many proofs, especially those in advanced undergraduate courses, require the use of formal notation. Students find many aspects of this notation, particularly the use of multiple quantifiers, to be troublesome. [Selden and Selden](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Sel2) (1995) asked 61 students in introductory proof courses to translate informal mathematical statements into the language of predicate calculus. They found that students were successful at this task less than 10% of the time. For instance, not one of the 20 students asked could express the statement ’A function *f* is increasing on an interval I means that for any numbers *x*1 and *x*2in I, if *x*1 x2, then *f* (*x*1) f (*x*2) as a logical sentence. Other research has illustrated how extracting meaning from a quantified logical statement is a very difficult and complex process (e.g.,[Dubinsky, Eltermann, and Gong](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Du), 1988). Some students' reasoning about multiply quantified statements involves only the predicate part of the statement, while ignoring how the variables are quantified (e.g., [Pinto and Tall](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Pi), 1999).

**2.4 Socio-mathematical norms**

Consider a student who is writing a proof and wants to use the following fact: ’If *f* (*x*) = *x*(1 + *x*) , then *f* ‘(*x*) = 2*x* + 1. What type of justification would be expected of such a student? The answer is, it depends. If the student was in a traditional introductory calculus course, he or she might be expected to perform a mathematical calculation. In a reform-oriented calculus course, numerical approximations or an inspection of the graph of *f* might suffice. In an undergraduate real analysis course, an argument using the definition of derivative would probably be required. In an advanced graduate course, this fact would be considered so obvious that an explanation would be unnecessary.

[Yackel and Cobb](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Ya) (1996) have coined the term *sociomathematical norms* to discuss how environmental influences, such as students' textbooks, teachers' comments, and their feedback on assignments, determine students' mathematical beliefs and subsequent behavior. [Dreyfus](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Dr) (1999) claims that ’What counts as an acceptable mathematical justification is one example of a sociomathematical norm. As illustrated above, at different times in a student's academic career, different types of justification are required. However, which type of justification is required is rarely explicated to the student. It is often the case that the student receives mixed messages. For instance, many mathematical textbooks will offer an intuitive explanation of one statement, an example to justify another statement, and a rigorous (formal) proof of another, yet the transition between intuitive, empirical, and rigorous thought is not clearly marked ([Dreyfus](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Dr), 1999). Researchers suggest that this may lead students to acquire undesirable mathematical beliefs about rigor, explanation, and proof, and may partially explain why students will submit informal arguments as proofs in advanced courses (e.g.,[Dreyfus](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Dr), 1999; [Raman](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Ra), 2002).

**2.5 Poor conceptual understanding and ineffective proof strategies**

Even if students are ’logically capable - that is, they know what constitutes a proof and they can reason deductively, recite and manipulate definitions, and draw valid inferences -- this does not guarantee that they can construct anything beyond very trivial proofs. Knowing logical rules and the definition of a concept does not ensure that students can reason about that concept. Students often require an intuitive (conceptual) understanding of the concept that they are working with before they can construct proofs. [Moore](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Mo) (1994) closely observed five undergraduates as they progressed through an introductory proof course. He found that students could sometimes state a concept's definition while having little understanding of the concept. These students could not, for instance, describe the concept in their own words or generate a single example of this concept. When these students were asked to write proofs about this concept, they did not know how to begin.

When writing a proof, there are many valid inferences one could draw. Clearly one is unlikely to construct a proof by deriving inferences in a haphazard manner. To illustrate, [Newell and Simon](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#NE) (1972) demonstrated how a breadth-first automated theorem prover would need to examine over 101000 proofs to prove some of the theorems in [Whitehead and Russell's](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Wh) classic logic text, *Principia Mathematica* (1935). In order to construct non-trivial proofs, undergraduates need strategies and heuristics to help them to decide how they should attack problems. In two studies, [Weber](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#We1) (2001, [2002b](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#We3)) observed eight undergraduates who had completed an abstract algebra course constructing non-trivial proofs about group homomorphisms and isomorphisms. The studies considered only those cases in which the undergraduates were aware of the facts and theorems needed to prove a statement and could construct a proof when specifically told which facts to use. Even in these cases, the undergraduates failed to construct a proof without prompting 68% of the time. Examination of these undergraduates' behaviors revealed that their strategies for constructing proofs were ineffective and crude. For instance, to prove a statement B, these undergraduates would often try to find *any* theorem of the form ’A implies B and try to prove A, even when the antecedent was implausible.

**How can the concept of proof be taught effectively?**

**3.1 The modified Moore method**

The modified Moore method (also called Texas-style instruction) is a teaching paradigm that is based on the pedagogical techniques of the mathematician R. L. Moore. Moore and proponents of this method believe that students will learn little about advanced mathematics by passively writing down the proofs that the professor presents on the blackboard, and will learn far more about mathematical concepts and proofs if they try to construct the proofs themselves. Below is a brief description of this influential teaching method.

In a typical class using the Moore method, the instructor presents the students with the definitions of mathematical concepts and perhaps a few motivating examples of those concepts. After this, students are asked to prove or disprove a set of propositions about these concepts. When a student believes that he or she has proved a proposition, that student is invited to present his or her argument on the blackboard. The teacher and fellow students may critique the student's work, or ask the student to clarify his or her argument. If everyone (including the teacher) is convinced by the proof, the class moves on to another proposition.

If no student is successfully able to prove a theorem, the instructor may ask the students to prove a simpler proposition, put the proposition off to another day, or simply let the proposition go unproved. The instructor may also provide assistance, but the assistance should be the minimal amount necessary for the students to construct the proof. What is critical is that the instructor never provides the students with the actual proof of a proposition. All proofs are generated by the students themselves.

For the sake of brevity, there are important issues (e.g., grading) that are not discussed here. A more complete description of the Moore method is given in [Jones](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof%22%20%5Cl%20%22Jo) (1977). A recent article discussing how the Moore method can be used specifically in undergraduate mathematics is given by [Mahavier](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Mah) (1999).

**3.2 Scientific debate**

Alibert and his colleagues at Grenoble University in France note that many mathematics majors and future high school mathematics teachers view proof as pedantic and divorced from the rest of mathematics ([Alibert](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Ali1), 1988; [Alibert and Thomas](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Ali2), 1991). This, they suspect, is due to the context in which these students encounter proof. They believe that by establishing an environment in which students may see and experience first-hand what is necessary for them to convince others of the truth or falsehood of propositions, proof becomes an instrument of personal value which they will be happier to use (or teach) in the future ([Alibert and Thomas](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Ali2), 1991, p. 230).

They create such an environment in a novel real analysis course through the use of ’scientific debate. To use this method, the instructor first presents students with a mathematical situation.

The students are asked to put forth arguments to convince their classmates of the conjecture's truth or falsehood. Alibert claims that (by consistently using this strategy) as the semester progresses, the students recognize the need for precise definitions, clear arguments, and rigorous proofs as a means of deciding whether conjectures are true or false. Thus, the students come to see proof as an instrumental tool used to settle scientific debates. Alibert found that 75% of students responding to a questionnaire preferred the method of scientific debate to traditional instruction, while only 10% rejected the method as being inaccessible or not sufficiently organized ([Alibert](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Ali1), 1988).

**3.3 Marty's introductory proof course**

[Marty](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Marty1) (1986) believes that a shortcoming of introductory proof courses is that they focus on content (i.e. the meaning of the statements that are proven or the proof itself), rather than on techniques. To address this, Marty created an introductory proof course where proof techniques were emphasized and the importance of mathematical content was minimized. The following summarize the principles that Marty used in his course:

* The purpose of the course was to cover a set of proof techniques (e.g., direct proof, proof by contradiction). Each proof technique was focused on one at a time and in great detail.
* Students encountered each proof technique in a variety of mathematical settings.
* Little attention was paid to mathematical content. Content was covered only deep enough that meaningful theorems could be presented and proved.
* Students were asked to apply proof techniques in a variety of settings and to present their proofs on the board. The problems the students were asked to solve were basic and did not involve cleverness or tricks.
* Alternative proofs to an already proved statement were encouraged.

To illustrate the success of his methods empirically, Marty examined the future mathematical success of every student who completed the introductory proof course at his university over a ten-year period. Marty compared the students who received his instruction with students who were taught in a traditional lecture format. He found that students in his class were two to three times more likely to pass their subsequent course in real analysis and four times as likely to continue their studies of advanced mathematics ([Marty](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Marty2), 1990).

**Proof in the high school classroom**

One reason that university students find proof so difficult is that their experience with constructing proofs is typically limited to high school geometry ([Moore](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Mo), 1994). To rectify this, the [*NCTM Standards*](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#NC)(2000) recommend that proof be introduced early in the mathematics curriculum, arguing that ’reasoning and proof should be a consistent part of students' mathematical experience in pre-kindergarten through grade 12 (p. 56). Further, the *NCTM Standards* and others (e.g., [Schoenfeld](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Sc), 1994) argue that proof should not be seen as a distinct mathematical topic, but rather as a way of thinking that can be applied to any mathematical topic. Finally, the *NCTM Standards* argue that by the time students complete 12th grade, they should recognize proof as fundamental to mathematics, be comfortable with constructing proofs, and be able to determine whether a given argument is a proof.

Considering the difficulty that mathematics majors have with proof, these goals are certainly ambitious. To assess the plausibility of these goals,[Knuth](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Kn) (2002) interviewed 16 qualified in-service high school teachers, some with a master's degree, to investigate their conceptions of mathematical proof. His findings were that many of these teachers' beliefs about proof were rather naive. For instance, when asked if a valid proof could ever become invalid, six teachers answered that contradictory evidence to a statement would invalidate that statement's proof. When asked about the role of proof, only three teachers indicated that proofs could be used to explain why statements were true and none responded that proofs could be used to promote understanding. Knuth concluded that many of these teachers would be unable to effectively meet the *NCTM Standards*.

Knuth and others (e.g., [Alibert and Thomas](http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof#Ali2), 1991) have suggested that students' and teachers' conceptions about proof are likely formed during their undergraduate mathematics courses. If one wants to improve the proof abilities of high school students, perhaps the best place for mathematics educators and mathematicians to look is toward the proof-oriented college courses for pre-service mathematics teachers.

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**Directions:** Read the whole article thoroughly the article. Then, select seven (or more) questions and answer them to facilitate reflection. Submit your paper following this format: Letter-sized, Times New Roman-12, normal margin, single spaced, justified.

1. Is developing in students the skill of “proving” important? Explain.
2. Express your agreement or disagreement with this statement:

*Many mathematics educators believe that focusing exclusively on the logical nature of proof can be harmful to students' development* (Alibert and Thomas, 1991).

1. In one part of the article, the author cites how one of his students provided a proof in one of his classes. How must a teacher address this kind of response from a student? How would you help the student understand that induction does not necessarily constitute good reasoning for proving?
2. Harel & Sowder (1998) presented a complete taxonomy of students’ beliefs of geometric proofs. Which of these beliefs have you encountered with your students or with your peers? Describe instances when you have detected these beliefs to be possessed by students.
3. With your knowledge of van Hiele’s model of geometric thought, suggest how you can use this knowledge to develop in students the skills relevant to proving.
4. How is developing appropriate (or pleasant) sociomathematical norms important? Suggest at least two teacher practices that can lead to positive sociomathematical norms.
5. Moore (1994) found that students could sometimes state a concept's definition while having little understanding of the concept. These students could not, for instance, describe the concept in their own words or generate a single example of this concept. Are instances similar to these common in our high school classes? How should definitions be discussed in class in order for students to fully grasp these concepts?
6. The author enumerates several methods that can help in developing the concept of proof effectively. Which of these methods would you use in a classroom? Briefly discuss how you will go about using the method you chose.
7. Comment on this by providing reference to your experiences with your Geometry class in college:

*If one wants to improve the proof abilities of high school students, perhaps the best place for mathematics educators and mathematicians to look is toward the proof-oriented college courses for pre-service mathematics teachers.*

1. This online article was lifted from the website of the Mathematical Association of America

Weber, K. (2016, January 5). *Research Sampler 8: Students' Difficulties with Proofs*. Retrieved from Mathematical Association of America: http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof [↑](#footnote-ref-1)